Two-type Galton-Watson Trees

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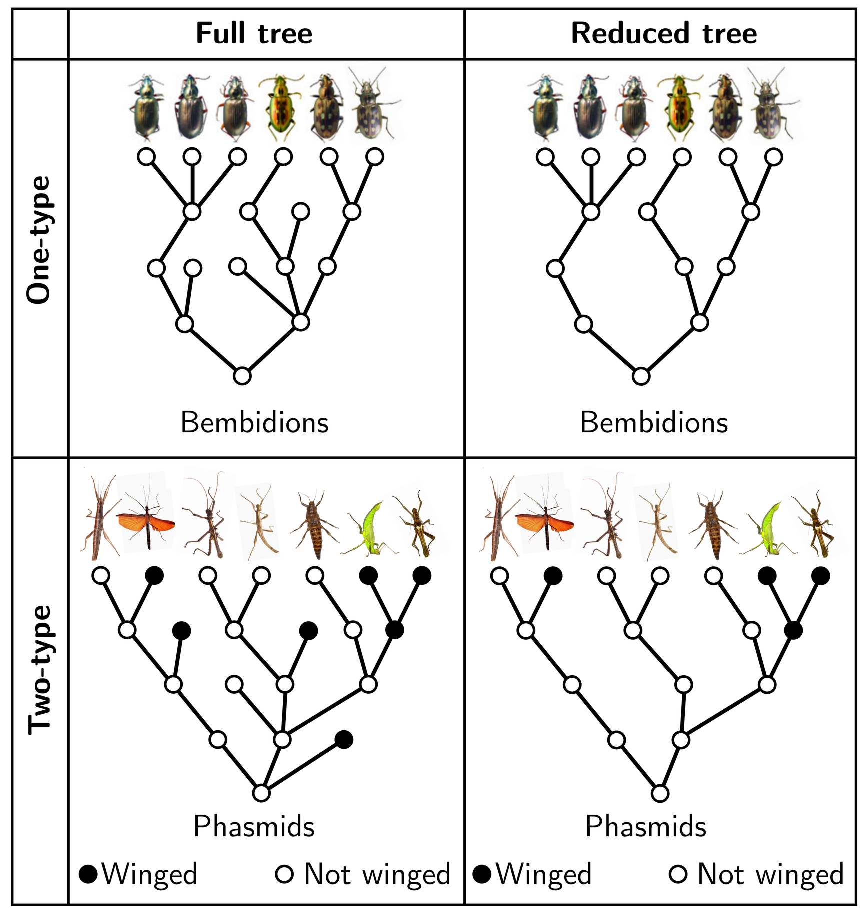
2013 CMS Summer Meeting
June 4 - 7, Halifax, Nova Scotia
Dalhousie University, Saint Mary's University

Motivation

Motivation and Goal

Generalizing some known constructions and lemmas concerning Galton-Watson trees (branching trees) to the multi-type case.

Galton-Watson Trees

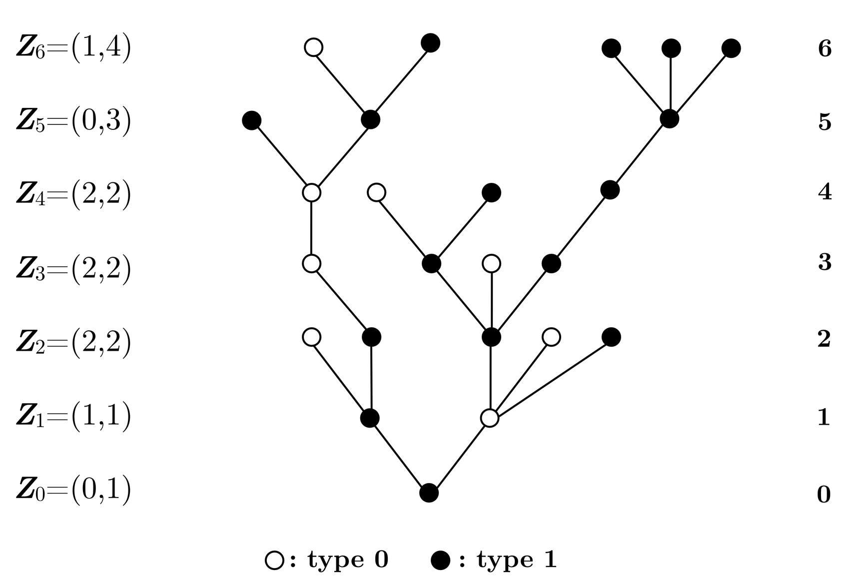


Galton-Watson trees are generated upwards according to the distribution of a random process.

Several algorithms exist to generate Galton-Watson trees backwards preserving their distribution; see [2], [4]. We aim to generalize the construction in [4] to the multi-type case.

Discrete Multi-type Galton-Watson Process

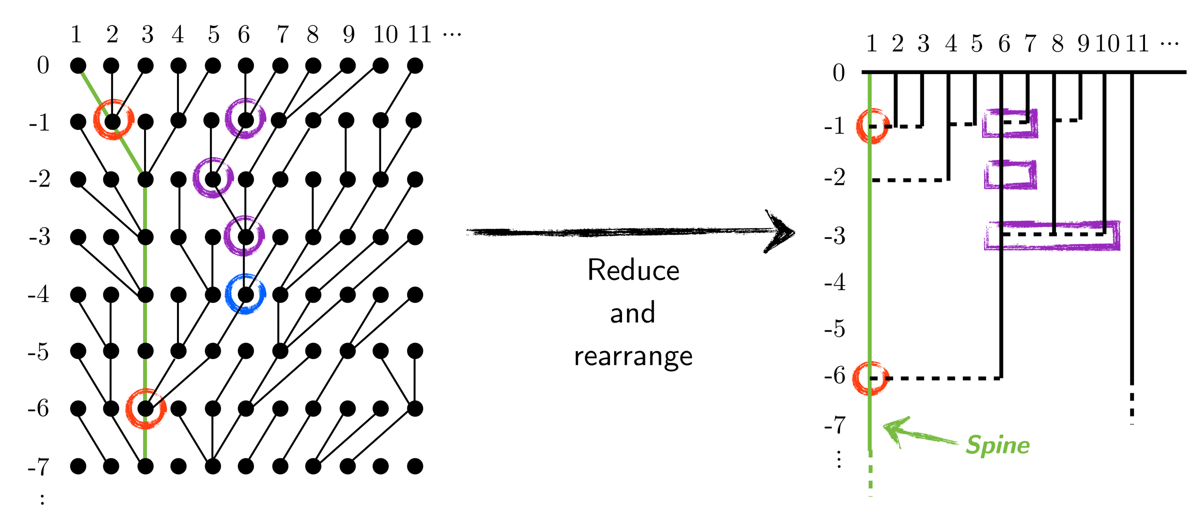
It is a Markov process $Z_n=(Z_n^{(0)},Z_n^{(1)},\dots,Z_n^{(k-1)})$, where $Z_n^{(i)}$, represents the number of individuals of type i at generation n in a G-W tree.



Coalescent Point Process

One-Type Coalescent Point Process

Doubly-infinite embedding of G-W tree / **Spine decomposition**



Notation

$a_i(n)$:= Horizontal index of the ancestor of individual $(0,i)$ in generation $-n$.

$$a_{10}(4) = 6$$

A_i := The coalescence time (most recent common ancestor) of individuals $(0,i)$ and $(0,i+1)$

$$A_1 = 1,\ A_2 = 1,\ \dots,\ A_5 = 6,\ \dots$$

$D_i(n)$:= number of children of individual $(-n,a_i(n))$ with descendants in $\{(0,j) : j \geq i\}$, not counting the lineage of $(0,i)$.

$$D_6(1) = 1,\ D_6(2) = 0,\ D_6(3) = 2,\ \dots$$

Coalescence Theorem, One-Type (Lambert and Popovic [4])

✤ $A_i = \min\{n \geq 1 \mid D_i(n) \neq 0\}$

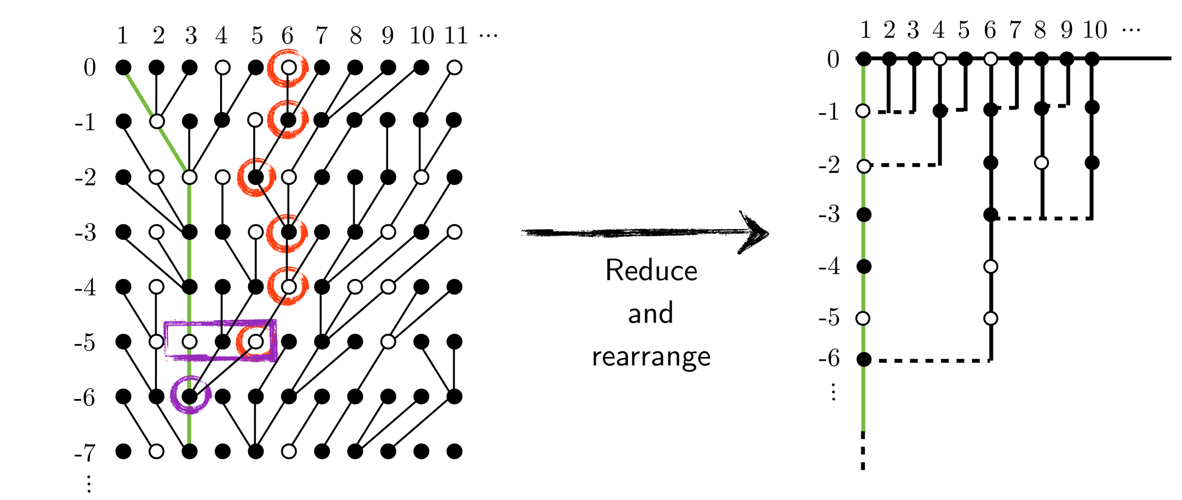
✤ The sequence $(D_i(\cdot))_{i\geq 0}$ is a Markov chain with transition probabilities:

$$(D_{i+1}(n) \,|\, D_i(\cdot) = d.) \overset{d}{=} \begin{cases} d_n & \text{for} \quad n > A_i, \\ d_{A_i} - 1 & \text{for} \quad n = A_i, \\ \zeta_n' & \text{for} \quad 1 \leq n < A_i, \end{cases}$$

where $\zeta_n' = (\zeta_n - 1 | \zeta_n \neq 0)$, and ζ_n is a binomial r.v with parameters (ξ, p_{n-1}) with ξ distributed as the number of offsprings of an individual in the G-W process and p_{n-1} being the probability of survival of this process at generation n.

Two-Type Coalescent Point Process

Doubly-infinite embedding of two-type G-W tree / **Spine decomposition**



Notation

$a_i(n)$, A_i, $D_i(n)$ are defined as for the one-type case.

$\boldsymbol{A}_i$:= Vector of types of individuals $(0,a_{i+1}(0))$, ... ,$(-(A_i-1),a_{i+1}(A_i-1))$.

$$\boldsymbol{A}_5 = (0,1,1,1,0,0),$$

$\boldsymbol{D}_i(n)$:= Vector of types of children of individual $(-n,\ a_i(n))$ having descendants of the form $(0,\ j)$ for $j \geq i$.

$$\boldsymbol{D}_5(6) = (0,0),$$
$$\text{length}(D_5(6)) - 1 = 1 = D_5(6)$$

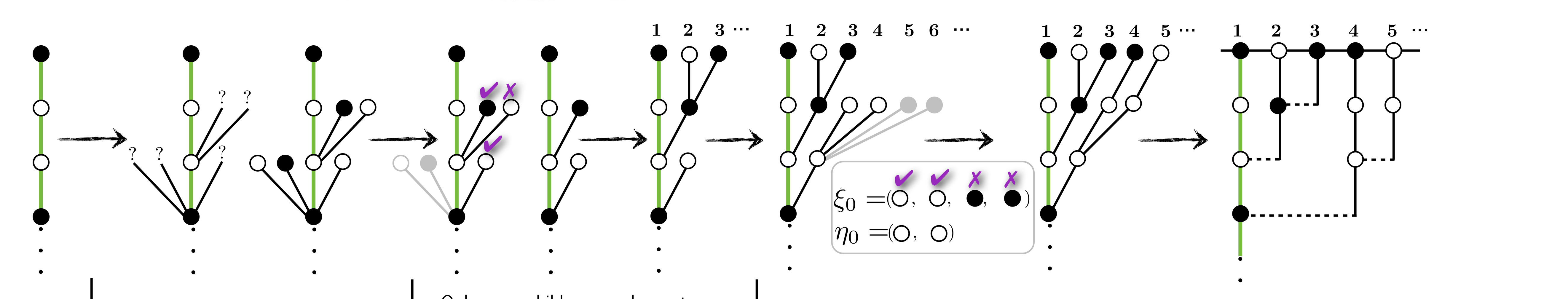
Coalescence Theorem, Two-Type (Rivas and Popovic, 2013)

Assuming fixed values for $\boldsymbol{D}_1(n)$ for $n \geq 0$, the sequence $(\boldsymbol{D}_i(\cdot))_{i\geq 0}$ is a Markov chain, with transition probabilities:

$$(\boldsymbol{D}_{i+1}(n) \,|\, \boldsymbol{D}_i(\cdot)) = \begin{cases} \boldsymbol{D}_i(n) \text{ for } n > A_i, \\ (\boldsymbol{D}_i(n)[2], \boldsymbol{D}_i(n)[3], \dots) \text{ for } n = A_i, \\ \boldsymbol{\eta}_{t_n}^n \text{ for all } n = A_i - 1, A_i - 2, \dots, 1, 0 \end{cases}$$

where $\boldsymbol{\eta}_{t_n}^n$ is a conditional r.v with certain known distribution. And $\boldsymbol{D}_i(n)[j]$ is the j-th coordinate of the vector $\boldsymbol{D}_i(n)$.

Backward construction of two-type G-W tree given its spine



Types of the spine are given.

Children of the individuals from the spine and their types are generated according to an extension of Geiger's lemma [2].

Only some children are chosen to survive in the spine decomposition, according to random variables with known distributions. This way $D_1(n)$ is obtained for all n.

The values of $D_i(n)$ for $i > 1$ are successively obtained from the Coalescence Theorem, Two-Type stated above, illustrating the Markov property.

Good news

✤ We generalize this construction to any number $k > 2$ of types.

✤ In the linear fractional case, additional results are obtained, such as a formula for the distribution of the A_i. More precisely, assume that offsprings are born according to a linear fractional distribution where the type of the first individual depends on the parent's type (given by a sub-stochastic matrix H) and the remaining offspring types are independent of the parent with a probability distribution $\boldsymbol{g}$. The pgf of the branching process in this case is given by

$$f_i(\boldsymbol{s}) = h_{i0} + \frac{\sum_{j=1}^k h_{ij} s_i}{1 + m - m\sum_{i=1}^k g_i s_i}$$
$$\boldsymbol{f}(\boldsymbol{s}) = (f_1(\boldsymbol{s}), f_2(\boldsymbol{s}), \dots, f_k(\boldsymbol{s}))$$

The A_i's turn out to be iid for this setting, with:

$$P(A_1 > n \,|\, \boldsymbol{l}_1) = P(A_1 > n) = \prod_{j=1}^{n} \frac{1}{1 + m - m\sum_{i=1}^k g_i h_{i0}^{(j-1)}}$$

where $h_{i0}{}^{(j-1)}$ represents the probability of a parent of type i having no children at generation $j-1$, obtained from $j-1$ iterations of $\boldsymbol{f}(\boldsymbol{s})$. The vector $\boldsymbol{l}_1$ is the one holding the types of the spine.

Additionally, we computed the distribution of B_i, which is defined as the time of the most recent common ancestor of individual $(0,i)$ and the leftmost individual to its right of the same type;

$$P(B_i > n \,|\, \boldsymbol{l}_i) = \prod_{j=1}^{n} \left(1 + m - m\sum_{y=1}^{k} g_y \left[h_{y0}^{(j-1)} + \frac{\sum_{z\neq t} h_{yz}^{(j-1)}}{1 + m^{(j-1)} - m^{(j-1)}\sum_{z\neq t} g_z^{(j-1)}}\right]\right)^{-1}$$

Future Work

✤ Find stationary distributions for the spine types in particular cases, in order to eliminate this initial requirement from the backward construction

References

[1]Athreya, Krishna B. and Ney, Peter E. Branching processes. Springer. New York. Dover Publications.

[2]Geiger, Jochen. Elementary new proofs of classical limit theorems for Galton-Watson processes. Journal of Applied Probability. Vol. 36(2). 1999.

[3]Lambert, Amaury. Population dynamics and random genealogies. Stochastic Models. Vol. 24(1). 2008.

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[5]Sagitov, Serik. Linear-fractional branching processes with countably many types. Stochastic processes and their applications. Vol. 123. 2013.

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Thanks to ISM.